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HOW BIG CAN THE CIRCUITS OF A BRIDGE OF A MAXIMAL CIRCUIT BE?

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If $C \subset E(G)$ is a maximum cardinality cocircuit of a 2-connected graph G, then no other maximum cocircuit is contained in one and the same block of $G-C$. The analogous conjecture for real representable matroids would have important applications to classifying convex bodies with a certain Helly type property.

0. Introduction

To classify the k-Helly dimensional convex bodies, it would be very important to know whether the following conjecture is true or not.

Conjecture 0.1. *Let $M(H)$ be a connected real linear matroid and $C \subseteqq H$ a maximum cardinality circuit of $M(H)$. Then no bridge of C in $M(H)$ — as a restriction of $M(H)$ — contains any $|C|$-element circuit.*

This problem has been solved so far only in some special cases. For example, it is true for graphic matroids (see [2]). In [3] it is proved for real linear matroids containing at most five element circuits. In this paper we prove the above statement for bond matroids of a graph.

1. Notations and Lemmas

$M(S)$ will denote a matroid on a set S, and $M^{\perp}(S)$ its dual matroid. If $C \subseteqq S$ then $M(S)/C$ is the contraction of $M(S)$ through C, and $M(S) \cdot C$ is the restriction of $M(S)$ to C. Let G be a simple graph. Then $B(G)$ denotes the bond matroid of G, and $P(G)$ the polygon matroid of G.

Definitions 1.1. Let $M(S)$ be a matroid. We say that $M(S)$ is *connected* if every pair of points of it is contained in a circuit. The maximal connected restrictions of $M(S)$ are called its *connected components.*

We define the *bridges* of $C \subseteqq M(S)$ as the connected components of $M(S)/C$.

We say that a subgraph G_1 of the graph G is a *block* of G if G_1 is connected, has no cut vertices and it is maximal with this properties.

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The following results can be found in [1].

Lemma 1.2. *Let $M(S)$ be a finite matroid. We have for $C \subseteqq S$, $(M(S)/C)^{\perp} = M^{\perp}(S) \times \times (S-C)$.* ∎

Lemma 1.3. *A finite matroid $M(S)$ is connected if and only if $M^{\perp}(S)$ is connected. The components of $M^{\perp}(S)$ are the dual matroids of the components of $M(S)$.* ∎

Lemma 1.4. *If G is a 1-connected graph, and G_1 and G_2 are vertex disjoint 1-connected subgraphs of G, then there exists a bond of G which separates G_1 and G_2.* ∎

Lemma 1.5. *The connected components of the polygon matroid of a graph G are the polygon matroids of the blocks of G.* ∎

2. The main result

Theorem 2.1. *Let $B(G)$ be the bond matroid of a simple graph G. If $B(G)$ is connected and C_0 is a maximum cardinality circuit in $B(G)$, then no bridge of C_0 — as a restriction of $B(G)$ — contains a $|C_0|$-element circuit.*

Proof. From Lemma 1.3 it follows that $B^{\perp}(G) = P(G)$ is connected, which means that G is 2-connected graph. Let C_0, $|C_0| = n$ be a maximal cardinality circuit in $B(G)$, that is, C_0 is an n-element bond of G. It is obvious from 1.3 that the bridges of C_0 in $B(G)$ are the dual matroids of the connected components of $(B(G)/C_0)^{\perp} = = P(G)\cdot(G-C_0)$. But using Lemma 1.5 we obtain that the connected components of $P(G)\cdot(G-C_0)$ are precisely the polygon matroids of the blocks of $G-C_0$.

Since C_0 is a bond of G, it separates G into two connected components, say K_1 and K_2, and so the blocks of the graph $G-C_0$ are the blocks of K_1 and K_2. Now let K_0 be a block of K_2, and suppose that there exists a bond C_1, $|C_1| = n$, of G with $C_1 \subseteqq K_0$. We distinguish two cases.

1st case. If C_1 is not bond of K_0. Let $K_{00}; K_{01}; \dots; K_{0l}$ be the 1-connected components of $K_0 - C_1$. Clearly $l \geqq 2$. We colour a vertex of K_0 red iff it is incident with one of the edges of C_0 or it is contained in an another block of K_2. It can be proved that any two red vertices can be connected by a path in $G-K_0$. This implies, since C_1 is a bond of G, that every K_{0i} containing red vertices, is in the same connected component of $G-C_1$, and those having no red vertices are in the other connected component. Thus there exists one and only one K_{0i}, say K_{00}, which has no red vertex, and there is no edge of C_1 connecting two distinct K_{0i}, K_{0j}, $1 \leqq i,\ j \leqq l,\ i \neq j$. Hence C_1 can be partitioned into l nonempty classes

$$C_{1i} = \{e_{ab} \in C_1 \colon\ a \in K_{00},\ b \in K_{0i}\}, \quad 1 \leqq i \leqq l.$$

The connected components of $K_2 - C_1$ are

$$K_{00},\ K_{0i} \cup T_i \quad 1 \leqq i \leqq l,$$

where T_i is the union of those connected components of $K_2 - K_0$ which have common vertex with K_{0i}. But then the family of sets

$$C_{0i} = \{e_{xy} \in C_0,\ x \in K_1,\ y \in K_{0i} \cup T_i\} \quad 1 \leqq i \leqq l$$

forms a partition of C_0.

Suppose that there exists an integer i_0, $1\leqq i_0\leqq l$, and $|C_{0i_0}|\neq|C_{1i_0}|$, say $|C_{0i_0}|<|C_{1i_0}|$. Then

$$C'=(\bigcup_{j\neq i_0} C_{0j})\cup C_{1i_0}$$

is a bond of G, and

$$|C'|=\sum_{j\neq i_0}|C_{0j}|+|C_{1i_0}|>\sum_{1\leqq j\leqq l}|C_{0j}|=n,$$

which is impossible. Hence for every i, $1\leqq i\leqq l$ $|C_{0i}|=|C_{1i}|$.

Lemma 2.2. *There exists a bond of K_{00}, which separates K_{00} into K_{001} and K_{002}, such that there is no C_{1i}, $1\leqq i\leqq l$, of which every edge is adjacent to K_{001}, and there exist at least two C_{1i}, C_{1j} $(i\neq j)$ both of which have at least one edge adjacent to K_{001}.*

Proof of the lemma. Let V_i denote the set of vertices of K_{00} incident with one of the edges of C_{1i}. Since K_0 is block, $|V_i|\geqq 2$, $1\leqq i\leqq l$. Now let S be a path in K_{00} connecting a vertex from V_1 to a vertex from V_2, such that S has no interior vertex belonging to V_1 or V_2. Let p be a vertex from V_1 and $p\notin S$. Using Lemma 1.4, it follows that there exists a bond $C_{00}^{(1)}$ of K_{00}, which separates K_{00} into $K_{001}^{(1)}$ and $K_{002}^{(1)}$, and $S\subseteq K_{001}^{(1)}$, $p\in K_{002}^{(1)}$. Suppose that for every j, $1\leqq j\leqq t$, we have a bond $C_{00}^{(j)}$ of K_{00} separating K_{00} into $K_{001}^{(j)}$ and $K_{002}^{(j)}$. We say that a vertex $p\in K_{001}^{(j)}$ is *green* iff there exists an integer $i(j)$, $1\leqq i(j)\leqq l$, and $p\in V_{i(j)}$, $V_{i(j)}\nsubseteq K_{001}^{(j)}$, and say it is *red* iff there exists an integer $i(j)$, $1\leqq i(j)\leqq l$ such that $p\in V_{i(j)}$, $V_{i(j)}\subseteq K_{001}^{(j)}$.

Assume that there exists a red-green path s' in $K_{001}^{(t)}$ (that is a path in $K_{001}^{(t)}$ connecting a red vertex to a green vertex), and a vertex $p\in K_{001}^{(t)}\cap C_{00}^{(t)}$, $p\notin s'$. Then, by Lemma 1.4, we obtain a bond C^* of $K_{001}^{(t)}$ that separates $K_{001}^{(t)}$ into $K_{0011}^{(t)}$ and $K_{0012}^{(t)}$ and $s'\subseteq K_{0011}^{(t)}$, $p\in K_{0012}^{(t)}$. But then $C_{00}^{(t)}$ can be divided into two classes:

$$C_{001}^{(t)}=\{e_{ab}\in C_{00}^{(t)}:\ a\in K_{002}^{(t)},\quad b\in K_{0011}^{(t)}\}$$

$$C_{002}^{(t)}=\{e_{ab}\in C_{00}^{(t)}:\ a\in K_{002}^{(t)},\quad b\in K_{0012}^{(t)}\}$$

($C_{001}^{(t)}$ may be empty). Clearly $C_{00}^{(t+1)}=C^*\cup C_{001}^{(t)}$ is a bond in K_{00}, and if it separates K_{00} into $K_{001}^{(t+1)}$ and $K_{002}^{(t+1)}$ then $s'\subseteq K_{001}^{(t+1)}$, $p\in K_{002}^{(t+1)}$. The bond sequence $C_{00}^{(i)}$ satisfies the following conditions.

(a) $K_{001}^{(i)}\supseteqq K_{001}^{(i+1)}$, $|V(K_{001}^{(i)})|>|V(K_{001}^{(i+1)})|$

(b) there is a green vertex in $K_{001}^{(1)}$, and if a vertex is green in $K_{001}^{(i)}$, then it is green in $K_{001}^{(i+1)}$ too.

(c) for every integer j there exist integers $i(j)$, $k(j)$,

$$i(j)\neq k(j),\quad 1\leqq i(j),\quad k(j)\leqq l,$$

and

$$V_{i(j)}\cap K_{001}^{(j)}\neq\emptyset,\quad V_{k(j)}\cap K_{001}^{(j)}\neq\emptyset.$$

Because of (a) the sequence is finite. Let $C_{00}^{(r)}$ be its last member. Then either $K_{001}^{(r)}$ contains no red vertices or every red-green path in $K_{001}^{(r)}$ contains all vertices of $C_{00}^{(r)}\cap K_{001}^{(r)}$.

In the first case, by (b) and (c), $C_{00}^{(r)}$ satisfies the conditions of the lemma. In the second case, the order of vertices of $C_{00}^{(r)}\cap K_{001}^{(r)}$ is the same on every red-green path in $K_{001}^{(r)}$. Let p_0 be the first in this order and s^* be a red-green r_0z_0 path in K_0.

We may suppose that s^* has no red interior vertex. Walk from r_0 on s^* till we are in $K_{001}^{(r)}$ and let q_0 be the last vertex (posibly $q_0=r_0$). Then q_0 is either green or $q_0\in K_{001}^{(r)}\cap C_{00}^{(r)}$.

If q_0 is green then the section r_0q_0 of s^* contains p_0. In the other case let r_1z_1 be a red-green path in $K_{001}^{(r)}$. If $p_0\notin r_0q_0$ then the union of r_0q_0 and the section q_0z_1 of r_1z_1 does not contain p_0 and it is connected, so there exists a red-green path in $K_{001}^{(r)}$ (between r_0 and z_1) not containing p_0, which is a contradiction. Hence we obtain that $p_0\in s^*$, and this means that p_0 is a cutpoint of K_0. But this is impossible since K_0 is block. This completes the proof of Lemma 2.2. ∎

Going on with the proof of the original theorem, let C_{00} be a bond in K_{00} as in Lemma 2.2. Suppose that $C_{11}, ..., C_{1i_0}$, $2\leqq i_0\leqq l$, are those which have a common vertex with both K_{001} and K_{002}, and $C_{1i_0+1}, ..., C_l$ are those which have not. Then we can write

$$C_{1j} = C_{1j}^1\cup C_{1j}^2, \quad 1\leqq j\leqq i_0,$$

where

$$C_{1j}^1 = \{e\in C_{1j}, \quad e \text{ is adjacent to } K_{001}\}$$

$$C_{1j}^2 = \{e\in C_{1j}, \quad e \text{ is adjacent to } K_{002}\}.$$

Since $|C_{11}^1|+|C_{11}^2|=|C_{11}|$, $|C_{12}^1|+|C_{12}^2|=|C_{12}|$, one of the inequalities $|C_{11}^2|+|C_{12}^1|\geqq\geqq|C_{11}|=|C_{01}|$ or $|C_{11}^1|+|C_{12}^2|\geqq|C_{12}|=|C_{02}|$ must be true. But

$$C' = C_{00}\cup C_{11}^2\cup C_{12}^1\cup\big(\bigcup_{3\leqq j\leqq i_0} C_{1j}^1\big)\cup\big(\bigcup_{\substack{1\leqq i\leqq l\\ i\neq 1}} C_{0i}\big)$$

$$C'' = C_{00}\cup C_{11}^1\cup C_{12}^2\cup\big(\bigcup_{3\leqq j\leqq i_0} C_{1j}^1\big)\cup\big(\bigcup_{\substack{1\leqq i\leqq l\\ i\neq 2}} C_{0i}\big)$$

are bonds of G, and

$$|C'|\geqq|C_{00}|+|C_{11}^2|+|C_{12}^1|+\sum_{\substack{1\leqq i\leqq l\\ i\neq 1}}|C_{0i}|\geqq 1+|C_{11}^2|+|C_{12}^1|+\sum_{\substack{1\leqq i\leqq l\\ i\neq 1}}|C_{0i}|$$

$$|C''|\geqq|C_{00}|+|C_{11}^1|+|C_{12}^2|+\sum_{\substack{1\leqq i\leqq l\\ i\neq 2}}|C_{0i}|\geqq 1+|C_{11}^1|+|C_{12}^2|+\sum_{\substack{1\leqq i\leqq l\\ i\neq 2}}|C_{0i}|.$$

Thus either C' or C'' has at least $n+1$ elements, which is impossible.

2nd case. C_1 is a bond of K_0. Then C_1 is bond of K_2, and it separates K_2 into K_{21} and K_{22}, and C_0 has common vertices only with K_{21}. Let s_1 and s_2 be two interior vertex disjoint paths between a vertex from K_1 and a vertex from K_{22}. Since $s_1\cap K_{21}$ and $s_2\cap K_{21}$ are vertex disjoint subgraphs of K_{21}, there exists a connected component s_1' of $s_1\cap K_{21}$ and a connected component s_2' of $s_2\cap K_{21}$ such that

$$s_1'\cap C_0\neq\emptyset, \quad s_1'\cap C_1\neq\emptyset$$

$$s_2'\cap C_0\neq\emptyset, \quad s_2'\cap C_1\neq\emptyset.$$

Hence, by Lemma 1.4, we obtain a bond C_2 of K_{21} that separates K_{21} into K_{21}^1 and K_{21}^2 such that $s_1'\subseteq K_{21}^1$, $s_2'\subseteq K_{21}^2$. Both C_0 and C_1 can be partitioned into two classes:

$$C_0 = C_0^1\cup C_0^2, \quad C_1 = C_1^1\cup C_1^2,$$

where

$$C_0^1 = \{e \in C_0,\ e \text{ is adjacent to } K_{21}^1\}$$

$$C_0^2 = \{e \in C_0,\ e \text{ is adjacent to } K_{21}^2\}$$

$$C_1^1 = \{e \in C_1,\ e \text{ is adjacent to } K_{21}^1\}$$

$$C_1^2 = \{e \in C_1,\ e \text{ is adjacent to } K_{21}^2\}.$$

Both $C_0^1 \cup C_2 \cup C_1^2$ and $C_0^2 \cup C_2 \cup C_1^1$ are bonds of G, and since

$$|C_0^1| + |C_1^2| + |C_0^2| + |C_1^1| = 2n,$$

one of them has at least $n+1$ elements, which is impossible. This completes the proof of the Theorem. ∎

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